

CERTAIN INEQUALITIES INVOLVING THE q -DEFORMED GAMMA FUNCTION[†]

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ABSTRACT. This paper is inspired by the work of J. Sándor in 2006. In the paper, the authors establish some double inequalities involving the ratio $\frac{\Gamma_q(x+1)}{\Gamma_q(x+\frac{1}{2})}$, where $\Gamma_q(x)$ is the q -deformation of the classical Gamma function denoted by $\Gamma(x)$. The method employed in presenting the results makes use of Jackson's q -integral representation of the q -deformed Gamma function. In addition, Hölder's inequality for the q -integral, as well as some basic analytical techniques involving the q -analogue of the psi function are used. As a consequence, q -analogues of the classical Wendel's asymptotic relation are obtained. At the end, sharpness of the inequalities established in this paper is investigated.

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1. INTRODUCTION AND PRELIMINARIES

Let $\Gamma(x)$ be the well-known classical Gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The psi function $\psi(x)$ otherwise known as the digamma function is defined as the logarithmic derivative of the Gamma function. That is,

$$\psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

The Jackson's q -integral from 0 to a and from 0 to ∞ are defined as follows [1]:

$$\int_0^a f(t) d_q t = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n,$$

$$\int_0^\infty f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n$$

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provided that the sums converge absolutely.

In a generic interval $[a, b]$, the Jackson's q -integral takes the following form:

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

For more information on this special integral, see [1].

For $a \in C$, the set of complex numbers, we have the following notations:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i) \quad \text{and} \quad [n]_q! = \frac{(q; q)_n}{(1-q)^n}.$$

The q -deformed Gamma function (also known as the q -Gamma function or the q -analogue of the Gamma function) is defined for $q \in (0, 1)$ and $x > 0$ by

$$\begin{aligned} \Gamma_q(x) &= \int_0^{\frac{1}{1-q}} t^{x-1} E_q^{-qt} d_q t = \int_0^{[\infty]_q} t^{x-1} E_q^{-qt} d_q t \\ &= (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+x}} \end{aligned} \quad (1)$$

where $E_q^t = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{t^n}{[n]_q!} = (-(1-q)t; q)_\infty$ is a q -analogue of the classical exponential function. See also [2], [3], [4], [5] and the references therein.

The function, Γ_q exhibits the following properties (see [3]),

$$\begin{aligned} \Gamma_q(x+1) &= [x]_q \Gamma_q(x), \\ \Gamma_q(1) &= 1, \\ \Gamma_q\left(\frac{1}{2}\right) &= \sqrt{\pi_q} \end{aligned} \quad (2)$$

where $[x]_q = \frac{1-q^x}{1-q}$, and $\pi_q = q^{\frac{1}{4}} \left(\left[-\frac{1}{2} \right]_{q^2}! \right)^2$ is the q -analogue of π . Note that π_q is obtained by setting $n = 0$ in the q -factorial, $[n]_q!$.

Let $\psi_q(x)$ be the q -analogue of the psi function similarly defined for $x > 0$ as follows:

$$\psi_q(x) = \frac{\Gamma'_q(x)}{\Gamma_q(x)} = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1 - q^{n+x}}$$

It is well-known in literature that this function is increasing for $x > 0$. For instance, see Lemma 2.2 of [6].

In 1987, Lew, Frauenthal and Keyfitz [7] by studying certain problems of traffic flow established the double inequality:

$$2\Gamma\left(n + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right) \Gamma(n+1) \leq 2^n \Gamma\left(n + \frac{1}{2}\right), \quad n \in N \quad (3)$$

The inequalities (3) can be rearranged as follows:

$$\frac{2}{\sqrt{\pi}} \leq \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \leq \frac{2^n}{\sqrt{\pi}}.$$

In 2006, Sándor [8] by using the following inequalities due Wendel [9]:

$$\left(\frac{x}{x+s}\right)^{1-s} \leq \frac{\Gamma(x+s)}{x^s \Gamma(x)} \leq 1 \quad (4)$$

for $x > 0$ and $s \in (0, 1)$, extended and refined inequality (3) as follows:

$$\sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \leq \sqrt{x+\frac{1}{2}} \quad (5)$$

for $x > 0$.

The objective of this paper is to establish certain inequalities involving the q -deformed Gamma function. First, employing similar techniques as in [8], [9], and [10], we prove an q -analogue of the double inequality (5). Next, using basic analytical procedures, we prove some related double inequality. At the end, we investigate the sharpness of the inequalities established.

2. MAIN RESULTS

Let us begin with the following Lemma.

Lemma 2.1. *Assume that $s \in (0, 1)$ and $q \in (0, 1)$. Then for any $x > 0$ the following inequality is valid.*

$$\left(\frac{[x]_q}{[x+s]_q}\right)^{1-s} \leq \frac{\Gamma_q(x+s)}{[x]_q^s \Gamma_q(x)} \leq 1 \quad (6)$$

Proof. We employ Hölder's inequality for Jackson's q -integral:

$$\int_0^\infty f(t)g(t) d_q t \leq \left[\int_0^\infty (f(t))^a d_q t \right]^{\frac{1}{a}} \left[\int_0^\infty (g(t))^b d_q t \right]^{\frac{1}{b}}$$

where $\frac{1}{a} + \frac{1}{b} = 1$ and $a > 1$.

Let $a = \frac{1}{1-s}$, $b = \frac{1}{s}$, $f(t) = t^{(1-s)(x-1)} E_q^{-(1-s)qt}$ and $g(t) = t^{sx} E_q^{-sqt}$

Then Hölder's inequality implies

$$\begin{aligned}
\Gamma_q(x+s) &= \int_0^{\frac{1}{1-q}} t^{x+s-1} E_q^{-qt} d_q t \\
&\leq \left[\int_0^{\frac{1}{1-q}} \left(t^{(1-s)(x-1)} E_q^{-(1-s)qt} \right)^{\frac{1}{1-s}} d_q t \right]^{1-s} \\
&\quad \times \left[\int_0^{\frac{1}{1-q}} \left(t^{sx} E_q^{-sqt} \right)^{\frac{1}{s}} d_q t \right]^s \\
&= \left[\int_0^{\frac{1}{1-q}} t^{x-1} E_q^{-qt} d_q t \right]^{1-s} \left[\int_0^{\frac{1}{1-q}} t^x E_q^{-qt} d_q t \right]^s \\
&= [\Gamma_q(x)]^{1-s} [\Gamma_q(x+1)]^s
\end{aligned}$$

Thus,

$$\Gamma_q(x+s) \leq [\Gamma_q(x)]^{1-s} [\Gamma_q(x+1)]^s \quad (7)$$

Substituting (2) into (7) yields

$$\Gamma_q(x+s) \leq [\Gamma_q(x)]^{1-s} [x]_q^s [\Gamma_q(x)]^s$$

which implies

$$\Gamma_q(x+s) \leq [x]_q^s \Gamma_q(x). \quad (8)$$

Replacing s by $1-s$ in equation (8) gives

$$\Gamma_q(x+1-s) \leq [x]_q^{1-s} \Gamma_q(x). \quad (9)$$

Substituting x by $x+s$ results to

$$\Gamma_q(x+1) \leq [x+s]_q^{1-s} \Gamma_q(x+s). \quad (10)$$

Now combining inequalities (8) and (10) gives

$$\frac{\Gamma_q(x+1)}{[x+s]_q^{1-s}} \leq \Gamma_q(x+s) \leq [x]_q^s \Gamma_q(x)$$

which can be written as

$$\frac{[x]_q}{[x+s]_q^{1-s}} \Gamma_q(x) \leq \Gamma_q(x+s) \leq [x]_q^s \Gamma_q(x). \quad (11)$$

Finally, equation (11) can be rearranged as:

$$\left(\frac{[x]_q}{[x+s]_q} \right)^{1-s} \leq \frac{\Gamma_q(x+s)}{[x]_q^s \Gamma_q(x)} \leq 1$$

concluding the proof of Lemma 2.1.

Theorem 2.2. Assume that $q \in (0, 1)$. Then the inequality

$$\sqrt{[x]_q} \leq \frac{\Gamma_q(x+1)}{\Gamma_q\left(x+\frac{1}{2}\right)} \leq \sqrt{\left[x+\frac{1}{2}\right]_q} \quad (12)$$

is valid for any $x > 0$.

Proof. By setting $s = \frac{1}{2}$ in the q -analogue of Wendel's inequalities (6), we get

$$\frac{1}{\sqrt{[x]_q}} \leq \frac{\Gamma_q(x)}{\Gamma_q\left(x + \frac{1}{2}\right)} \leq \frac{\sqrt{\left[x + \frac{1}{2}\right]_q}}{[x]_q} \quad (13)$$

Using (2), we can arrange (13) as follows:

$$\sqrt{[x]_q} \leq \frac{\Gamma_q(x+1)}{\Gamma_q\left(x + \frac{1}{2}\right)} \leq \sqrt{\left[x + \frac{1}{2}\right]_q}.$$

That completes the proof.

Remark 2.3. Inequalities (6) imply

$$\lim_{x \rightarrow \infty} \frac{\Gamma_q(x+s)}{[x]_q^s \Gamma_q(x)} = 1. \quad (14)$$

Remark 2.4. Since $[x]_q^{\beta-\alpha} \frac{\Gamma_q(x+\alpha)}{\Gamma_q(x+\beta)} = \frac{\Gamma_q(x+\alpha)}{[x]_q^\alpha \Gamma_q(x)} \cdot \frac{[x]_q^\beta \Gamma_q(x)}{\Gamma_q(x+\beta)}$, we obtain by (14),

$$\lim_{x \rightarrow \infty} [x]_q^{\beta-\alpha} \frac{\Gamma_q(x+\alpha)}{\Gamma_q(x+\beta)} = 1, \quad \alpha, \beta \in (0, 1). \quad (15)$$

Remark 2.5. The equalities (14) and (15) are the q -analogues of the classical Wendel's asymptotic relation [9]:

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+s)}{x^s \Gamma(x)} = 1. \quad (16)$$

Theorem 2.6. Assume that $q \in (0, 1)$ is fixed. Then the inequality

$$\frac{1}{\sqrt{\pi_q}} < \frac{\Gamma_q(x+1)}{\Gamma_q\left(x + \frac{1}{2}\right)} < (1 + \sqrt{q}) \cdot \frac{1}{\sqrt{\pi_q}} \quad (17)$$

is valid for $x \in (0, 1)$.

Proof. Define a function $U(q, x)$ for $q \in (0, 1)$ and $x \geq 0$ by

$$U(q, x) = \frac{\Gamma_q(x+1)}{\Gamma_q\left(x + \frac{1}{2}\right)}.$$

Notice that $\Gamma_q(1) = \Gamma_q(2) = 1$, $\Gamma_q\left(\frac{1}{2} + 1\right) = \left[\frac{1}{2}\right]_q \Gamma_q\left(\frac{1}{2}\right) = \left[\frac{1}{2}\right]_q \sqrt{\pi_q}$, $\left[\frac{1}{2}\right]_q = \frac{1-\sqrt{q}}{1-q}$, $U(q, 0) = \frac{1}{\sqrt{\pi_q}}$ and $U(q, 1) = (1 + \sqrt{q}) \cdot \frac{1}{\sqrt{\pi_q}}$.

Now let $f(q, x) = \ln U(q, x)$. Then,

$$f(q, x) = \ln \frac{\Gamma_q(x+1)}{\Gamma_q\left(x + \frac{1}{2}\right)} = \ln \Gamma_q(x+1) - \ln \Gamma_q\left(x + \frac{1}{2}\right).$$

For a fixed $q \in (0, 1)$ we obtain

$$f'(q, x) = \psi_q(x+1) - \psi_q\left(x + \frac{1}{2}\right) > 0,$$

since $\psi_q(x)$ is increasing for $x > 0$. Hence, $U(q, x) = e^{f(q, x)}$ is increasing on $x > 0$ and for $x \in (0, 1)$ we have

$$U(q, 0) < U(q, x) < U(q, 1)$$

establishing (17).

Remark 2.7. Define F by $F(q, x) = [x]_q^{-\frac{1}{2}} \frac{\Gamma_q(x+1)}{\Gamma_q(x+\frac{1}{2})}$ for $q \in (0, 1)$ and $x > 0$. Let $g(x) = \ln F(q, x) = \ln \Gamma_q(x+1) - \ln \Gamma_q(x+\frac{1}{2}) - \frac{1}{2} \ln [x]_q$. Then, $g'(x) = \psi_q(x+1) - \psi_q(x+\frac{1}{2}) + \frac{1}{2} \frac{q^x \ln q}{1-q^x}$. The following Stieltjes integral representations are valid.

$$\psi_q(x) = -\ln(1-q) - \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} d\mu_q(t), \quad \int_0^\infty e^{-xt} d\mu_q(t) = -\frac{q^x \ln q}{1-q^x}$$

where $\mu_q(t) = -\ln q \sum_{k=1}^\infty \delta(t+k \ln q)$ and δ represents the Dirac delta function. See [11] and the references therein. Then that implies,

$$\begin{aligned} g'(x) &= - \int_0^\infty \frac{e^{-(x+1)t}}{1-e^{-t}} d\mu_q(t) + \int_0^\infty \frac{e^{-(x+\frac{1}{2})t}}{1-e^{-t}} d\mu_q(t) - \frac{1}{2} \int_0^\infty e^{-xt} d\mu_q(t) \\ &= \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} \phi(t) d\mu_q(t) \end{aligned}$$

where $\phi(t) = e^{-\frac{1}{2}t} - \frac{1}{2}e^{-t} - \frac{1}{2} < 0$. By the Hausdorff-Bernstein-Widder theorem (see [12] and the references therein), we obtain $g'(x) < 0$, so $g(x)$ is strictly decreasing. Consequently, $F(q, x)$ is strictly decreasing. Hence, $F(q, x) \geq \lim_{x \rightarrow \infty} F(q, x) = 1$ yielding the lower bound of (12).

Remark 2.8. Define G by $G(q, x) = [x + \frac{1}{2}]_q^{-\frac{1}{2}} \frac{\Gamma_q(x+1)}{\Gamma_q(x+\frac{1}{2})}$ for $q \in (0, 1)$ and $x > 0$. Let $w(x) = \ln G(q, x) = \ln \Gamma_q(x+1) - \ln \Gamma_q(x+\frac{1}{2}) - \frac{1}{2} \ln [x + \frac{1}{2}]_q$. Then, $w'(x) = \psi_q(x+1) - \psi_q(x+\frac{1}{2}) + \frac{1}{2} \frac{q^{x+\frac{1}{2}} \ln q}{1-q^{x+\frac{1}{2}}}$. By setting $a = \frac{1}{2}$, $b = 1$, $c = \frac{1}{2}$ and $k = 1$ in Theorem 7.2 of [13], we obtain,

$$\psi_q(x+1) - \psi_q(x+\frac{1}{2}) \geq -\frac{1}{2} \frac{q^{x+\frac{1}{2}} \ln q}{1-q^{x+\frac{1}{2}}}.$$

That implies $w'(x) \geq 0$, so $w(x)$ is increasing. As a result, $G(q, x)$ is also increasing. Hence, $G(q, x) \leq \lim_{x \rightarrow \infty} G(q, x) = 1$ yielding the upper bound of (12).

Remark 2.9. Let $H(q, x) = \frac{\sqrt{\pi q} \Gamma_q(x+1)}{\Gamma_q(x+\frac{1}{2})}$. Then, $H(q, x)$ is increasing, and for $x \in (0, 1)$ we have,

$1 = \lim_{x \rightarrow 0^+} H(q, x) \leq H(q, x)$ and $H(q, x) \leq \lim_{x \rightarrow 1^-} H(q, x) = 1 + \sqrt{q}$ respectively yielding the lower and upper bounds of (17).

By the above remarks, the estimates in (12) and (17) are sharp.

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